Boolean Arithmetic

Counting systems

quantity	decimal	binary	3-bit register
	0	0	000
*	1	1	001
**	2	10	010
***	3	11	011
****	4	100	100
*****	5	101	101
******	6	110	110
*******	7	111	111
********	8	1000	overflow
*********	9	1001	overflow
**********	10	1010	overflow

Rationale

$$(9038)_{ten} = 9 \cdot 10^3 + 0 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0 = 9038$$

$$(10011)_{two} = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 19$$

$$(x_n x_{n-1} \ldots x_0)_b = \sum_{i=0}^{n} x_i \cdot b^i$$

Binary addition

- Assuming a 4-bit system:

0	0	0	1			1	1	1	1	
	1	0	0	1			1	0	1	1
	0	1	0	1			0	1	1	1
0	1	1	1	0		1	0	0	1	0

no overflow — overflow

- Algorithm: exactly the same as in decimal addition
- Overflow (MSB carry) has to be dealt with.

Representing negative numbers (4-bit system)

0	0000		
1	0001	1111	-1
2	0010	1110	-2
3	0011	1101	-3
4	0100	1100	-4
5	0101	1011	-5
6	0110	1010	-6
7	0111	1001	-7
		1000	-8

- The codes of all positive numbers begin with a "0"
- The codes of all negative numbers begin with a "1"
- To convert a number:
  leave all trailing 0's and first 1 intact,
  and flip all the remaining bits

Example: 2 - 5 = 2 + (-5) =

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  0 0 1 0
+ 1 0 1 1
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  1 1 0 1   =  -3
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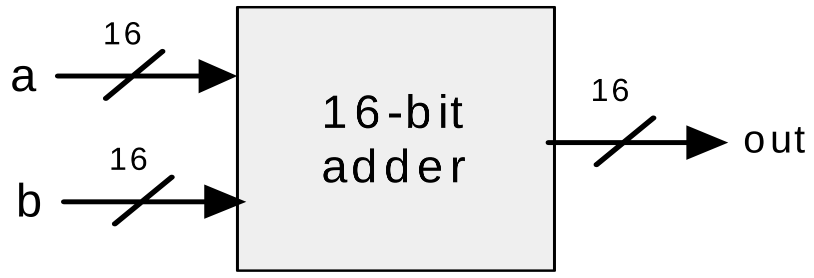
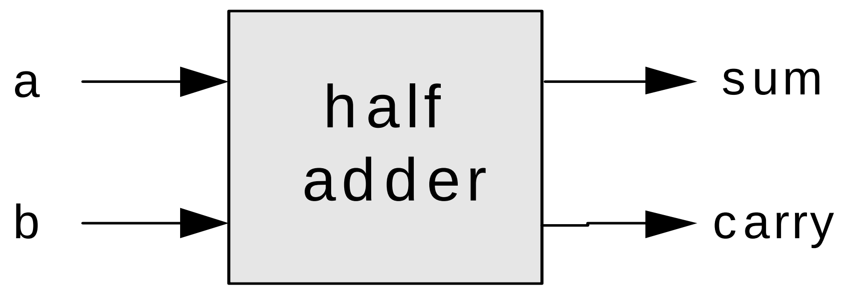
Building an Adder chip



- Adder: a chip designed to add two integers
- Proposed implementation:
  - Half adder: designed to add 2 bits
  - Full adder: designed to add 3 bits
  - Adder: designed to add two *n*-bit numbers.

Half adder (designed to add 2 bits)

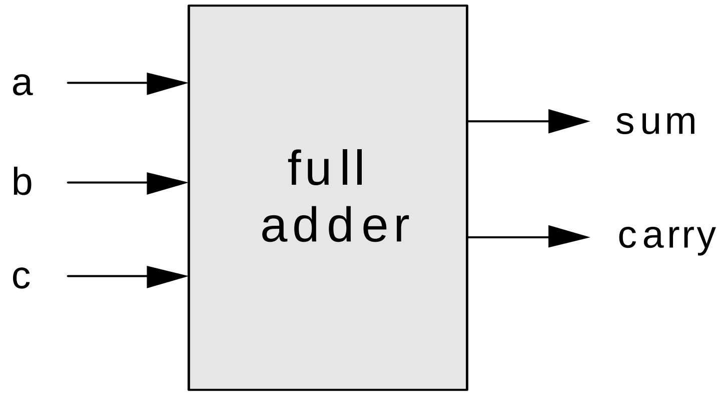
a	b	carry	sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0



- Implementation: based on two gates that you've seen before.

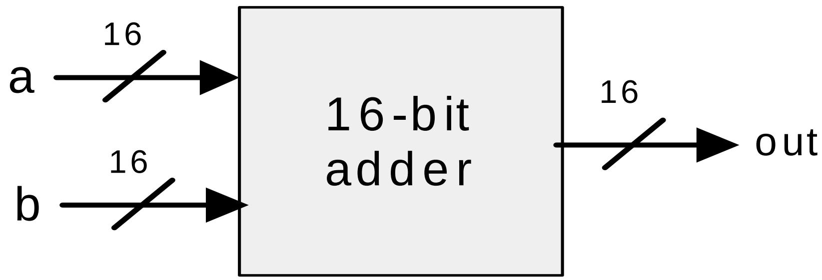
Full adder (designed to add 3 bits)

a	b	c	carry	sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



- Implementation: can be based on half-adder gates.

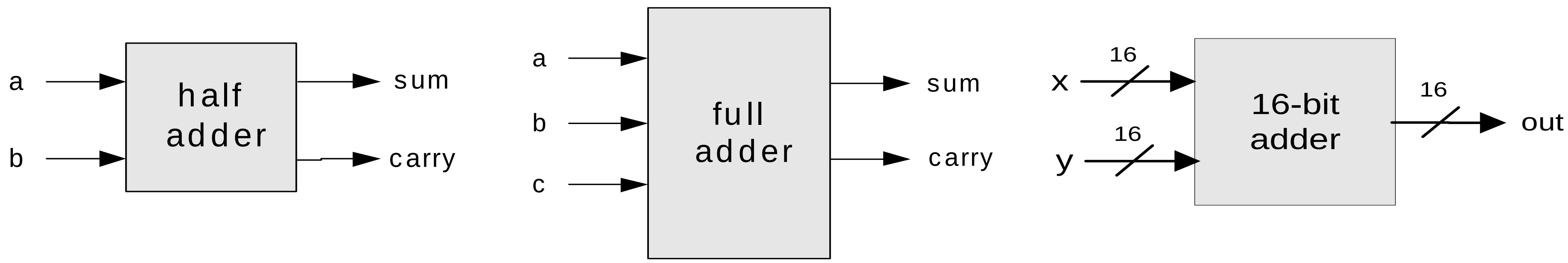
n-bit Adder

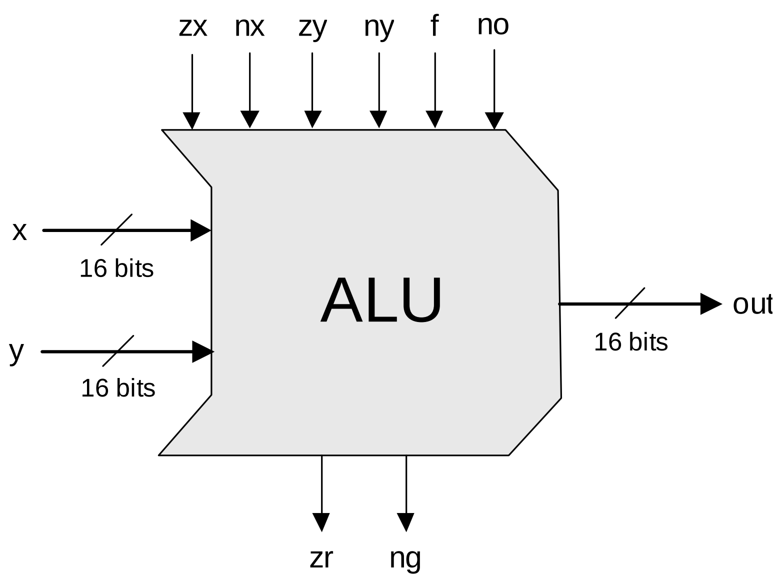


...	1	0	1	1	a	
						+
...	0	0	1	0	b	
...	1	1	0	1	out	

- Implementation: array of full-adder gates.

The ALU (of the Hack platform)





out(x, y, control bits**) =**

x+y, x-y, y–x,

0, 1, -1,

x, y, -x, -y,

x!, y!,

x+1, y+1, x-1, y-1,

x&y, x|y

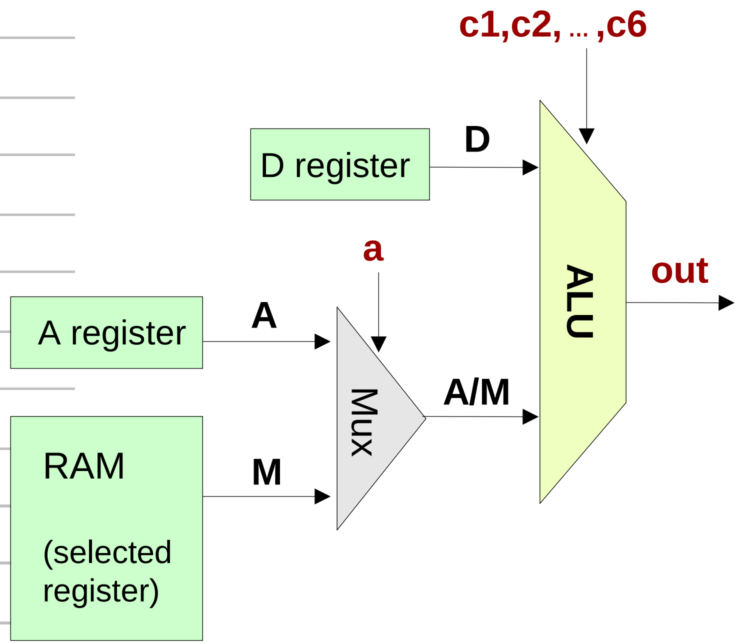
ALU logic (Hack platform)

These bits instruct how to pre-set the x input		These bits instruct how to pre-set the y input		This bit selects between + / And	This bit inst. how to post-set out	Resulting ALU output
zx	nx	zy	ny	f	no	out=
if zx then x=0	if nx then x=!x	if zy then y=0	if ny then y=!y	if f then out=x+y else out=x And y	if no then out=!out	f(x,y)=
1	0	1	0	1	0	0
1	1	1	1	1	1	1
1	1	1	0	1	0	-1
0	0	1	1	0	0	x
1						y
0						!x
1						!y
0						-x
1						-y
0	1	1	1	1	1	x+1
1	1	0	1	1	1	y+1
0	0	1	1	1	0	x-1
1	1	0	0	1	0	y-1
0	0	0	0	1	0	x+y
0	1	0	0	1	1	x-y
0	0	0	1	1	1	y-x
0	0	0	0	0	0	x&y
0	1	0	1	0	1	x|y

<u>Implementation</u>: build a logic gate architecture that “reads” each control bit and does what the table specifies: if zx=1 then set x to 0, etc.

The ALU in the CPU context (Hack platform)

out (when a=0)	c1	c2	c3	c4	c5	c6	out (when a=1)
0	1	0	1	0	1	0	
1	1	1	1	1	1	1	
-1	1	1	1	0	1	0	
D	0	0	1	1	0	0	
A	1	1	0	0	0	0	M
!D	0	0	1	1	0	1	
!A	1	1	0	0	0	1	!M
-D	0	0	1	1	1	1	
-A	1	1	0	0	1	1	-M
D+1	0	1	1	1	1	1	
A+1	1	1	0	1	1	1	M+1
D-1	0	0	1	1	1	0	
A-1	1	1	0	0	1	0	M-1
D+A	0	0	0	0	1	0	D+M
D-A	0	1	0	0	1	1	D-M
A-D	0	0	0	1	1	1	M-D
D&A	0	0	0	0	0	0	D&M
D|A	0	1	0	1	0	1	D|M



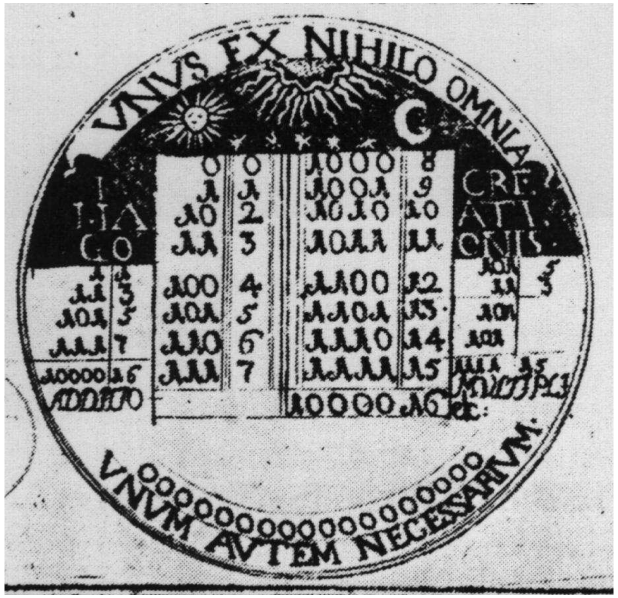
Perspective

- Combinational logic
- Our adder design is very basic: no parallelism
- It pays to optimize adders
- Our ALU is also very basic: no multiplication / division
- Where is the seat of advanced math operations?
  a typical hardware/software tradeoff.

Historical note: Leibnitz (1646-1716)

- "The binary system may be used in place of the decimal system; express all numbers by unity and by nothing"
- 1679: built a mechanical calculator (+, -, *, /)

- CHALLENGE: "All who are occupied with the reading or writing of scientific literature have assuredly very often felt the want of a common scientific language, and regretted the great loss of time and trouble caused by the multiplicity of languages employed in scientific literature:
- SOLUTION: *"Characteristica Universalis"*: a universal, formal, language of reasoning
- The dream's end: Turing and Goedl in 1930's.



Leibniz's medallion for the Duke of Brunswick